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A HEAT TRANSFER PREDICTION METHOD FOR TURBULENT BOUNDARY LAYERS DEVELOPING OVER ROUGH SURFACES WITH TRANSPIRATION

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NOMENCLATURE

A, smooth-wall sublayer thickness;
A^+, AU_τ/ν;
A_R, rough-wall sublayer thickness;
A_R^+, $A_R U_\tau/\nu$;
$C_f/2$, local skin friction coefficient;
C_p, specific heat of fluid;
D, pipe diameter;
F, blowing fraction;
k_s, equivalent sandgrain roughness;
K_R, fully rough acceleration parameter, $(r/U_\infty)/(\mathrm{d}U_\infty/\mathrm{d}x)$;
l, mixing length;
l^+, lU_τ/ν;
Pr, molecular Prandtl number;
$\dot{q}''_w$, wall heat flux;
r, radius of spheres comprising test surface;
R, pipe radius;
Re_k, roughness Reynolds number, $k_s U_\tau/\nu$;
Re_D, pipe diameter Reynolds number;
Re_{Δ_2}, enthalpy thickness Reynolds number;
St, Stanton number;
δt_0, fully rough wall temperature step;
$(\delta t_0)^+$, $\delta t_0/T_\tau$;
T, mean temperature;
T_w, wall temperature;
T_τ, $\dot{q}''_0/\rho C_p U_\tau$;
U_∞, freestream velocity;
U_τ, friction velocity;
V_0, velocity of transpired fluid at the wall;
V_0^+, V_0/U_τ;
x, coordinate in downstream direction;
y, coordinate normal to surface;
y^+, yU_τ/ν;
δ_2, momentum thickness;
κ, Karman constant;
ν, kinematic viscosity;
ρ, density.

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INTRODUCTION

THE PURPOSE of the present communication is to present a closure method for the boundary layer equations which can be used to predict Stanton numbers, skin friction coefficients, and mean profiles in boundary layers developing over rough surfaces. The method is the only published one known to the authors for which the combined effects of heat and momentum transfer with both favourable pressure gradient and transpiration may be predicted. Closure is accomplished by specification of mixing-length and turbulent Prandtl number distributions, along with a wall temperature step.

Techniques presently available to predict effects of roughness on turbulent flows are numerous. One of the earliest of these incorporating a mixing-length closure was suggested by van Driest [1]. More recently developed methods range from the integral techniques of Dvorak [2, 3] to differential boundary layer methods such as that suggested by Antonia and Wood [4]. Another recent technique is presented by Cebeci and Chang [5], who discuss a differential method with near wall mixing-length equations based on contributions by

Rotta [6]. McDonald and Fish [7] use a correction term which accounts for roughness effects in a mixing-length model used to predict transition between laminar and turbulent flow. Finson [8] also predicts boundary-layer transition over rough surfaces, but he uses a Reynolds stress closure model. Another high order closure model is presented by Adams and Hodge [9], who use an integral form of the turbulent kinetic energy equation with a term added to represent the generation of turbulence which occurs in the wakes behind roughness elements. Hatton and Walklate [10] and Wassel and Mills [11] suggest mixing-length models for heat transfer in pipes. For transpiration predictions, Schetz and Nerney [12] developed methods to predict hydrodynamic boundary layers developing over rough surfaces using experimental data to extend models described by van Driest and by Reichardt. Healzer, Moffat and Kays [13] present a method which allows prediction of heat transfer and transpiration in boundary layers similar to those discussed in this work. However, aspects of the Healzer, Moffat and Kays closure model are not physically plausible, and the scheme is not useful for all of the experimental cases predicted using the present model.

PREDICTION MODEL

In boundary layers developing over rough surfaces, positive transpiration causes the flow to behave as if the roughness of the surface is greater than is actually present. This effect is evident from the increases in near wall turbulence levels from transpiration, which result in a less favourable environment for the presence of a viscous sublayer. Transpiration has a similar effect on boundary layers developing over smooth surfaces, since a primary result of blowing is a reduction in effective viscous sublayer thickness. The increase in near-wall mixing-length from transpiration through rough walls may then be accounted for by increasing the effective roughness Reynolds number using

$$\overline{Re}_k = Re_k\left(1 + 16eV_0^+\right) \tag{1}$$

where the function e is empirical with the values $e = 1.0$ for $Re_k \geq Re_{k,R}$ and $e = Re_k/Re_{k,R}$ for $Re_k < Re_{k,R}$. $Re_{k,R}$ is the value of the roughness Reynolds number which separates fully rough from transitionally rough behaviour.

Following the development of Ligrani, Moffat and Kays [14] the near-wall mixing-length equation used for fully rough boundary layers with and without transpiration, and with and without favourable pressure gradients is then given as

$$l^+ = \kappa\left(y^+ + 0.0307\,(\overline{Re}_k - Re_k')\right). \tag{2}$$

Expressing Re_k' as $k_s U_\tau'/\nu$, and the mixing-length in equation (2) may be expressed without any explicit viscosity dependence as

$$l = \kappa\left[y + 0.0307\,k_s\left(1 + 16e\frac{V_0}{U_\tau} - \frac{U_\tau'}{U_\tau}\right)\right]. \tag{3}$$

The U_τ'/U_τ term in (3) then provides an implicit viscosity dependence for the fully rough mixing-length since viscosity influences the value of U_τ at $Re_k = Re_k'$. No viscous sublayer exists in fully rough flows, however, viscosity may not have a completely negligible effect on fully rough hydrodynamic behaviour for roughness Reynolds numbers ranging from $Re_{k,R}$ to about 200. In equations (1)–(3), Re_k' and $Re_{k,R}$ are constant for a given type of roughness, and are set equal to 46.0 and 55.0, respectively, for all roughness types considered in the present work. Equations (2) and (3) were developed by first considering that form drag on roughness elements may be represented using a non-zero value of wall mixing-length. The dependence of this wall mixing-length offset on k_s was then determined by equating the velocity profile from this mixing-length to the velocity profile equation from experiments.

When Re_k is less than $Re_{k,R}$ transitionally rough behaviour exists and a viscous sublayer is present. The thickness of this viscous sublayer is less than the smooth wall value, but greater than the value for fully rough flows, where the sublayer is considered to be destroyed. The viscous sublayer thickness A_R^+ then decreases in effective magnitude as Re_k increases above $Re_{k,s}$ the roughness Reynolds number which separates transitionally rough behaviour from smooth-wall behaviour. The mixing-length for such flows may be modelled using the van Driest equation

$$l^+ = ky^+\left(1 - e^{-y^+/A_R^+}\right) \tag{4}$$

where the functional dependence of A_R^+ on $\overline{Re}_k$ is given by

$$A_R^+ = A^+ f \tag{5}$$

with

$$f = \frac{ln\,(Re_{k,R}) - ln\,(\overline{Re}_k)}{ln\,(Re_{k,R}) - ln\,(Re_{k,s})}$$

$$\text{for } Re_{k,s} \leqslant \overline{Re}_k \leqslant Re_{k,R} \tag{6a}$$

$$f = 0 \quad \text{for } \overline{Re}_k > Re_{k,R} \tag{6b}$$

and

$$f = 1 \quad \text{for } \overline{Re}_k < Re_{k,s}. \tag{6c}$$

$Re_{k,s}$ is strongly dependent on roughness geometry and is set

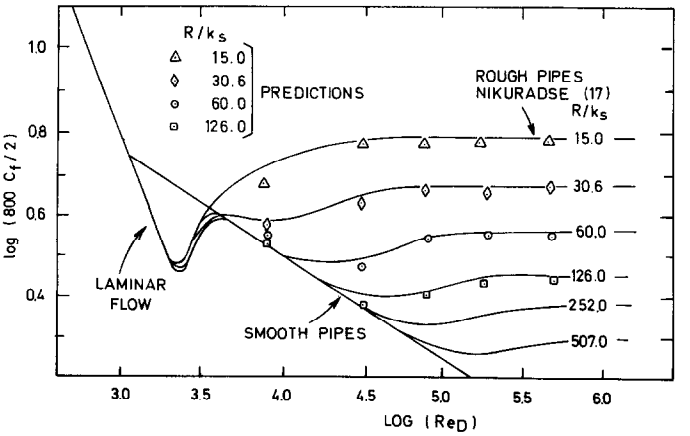

FIG. 1. Prediction of Nikuradse's [17] pipe skin friction coefficient data.

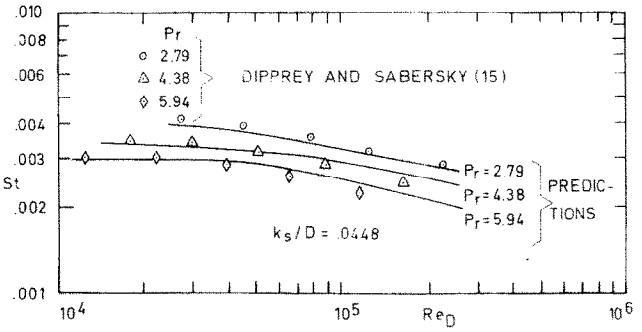

FIG. 2. Prediction of Dipprey and Sabersky's [15] pipe heat transfer data.

equal to 7.00, and 15.0 for flows over sandgrain and uniform packed spheres roughness, respectively.

When heat transfer is present in turbulent boundary layers which are fully rough, a conduction sublayer is present which can be described as a very thin film of fluid surrounding the roughness elements, where heat transfer is principally by molecular conduction. As the flow becomes transitionally rough and smooth, the conduction sublayer becomes less important as a viscous sublayer begins to surround the areas between and above roughness elements. In the present prediction method, the conduction sublayer is modelled using a temperature step at the wall given by

$$(\delta t_0)^+ = (1 - f)\, k_f' (\overline{Re}_k)^{0.20} (Pr)^{0.44} \qquad (7)$$

where $k_f' = 1.00$ for uniform spheres roughness, and $k_f' = 2.86$ for close-packed sandgrain type roughness. For $Re_k > Re_{k,R}$, equation (7) is equivalent to an equation suggested by Dipprey and Sabersky [15], where the non-dimensionalized mean roughness height in the original equation is replaced by $\overline{Re}_k$ in this study. The turbulent Prandtl number distribution used for the present predictions is the same as that suggested by Crawford and Kays [16]. However, a constant tubulent Prandtl number of 0.90 may also be used instead of this equation to give equally good results.

Additional discussion of the physical arguments and derivations of the equations used in the present closure model are presented by Ligrani, Moffat and Kays [14]. For predictions, the closure equations were incorporated into an existing turbulent boundary layer prediction scheme described by Crawford and Kays [16].

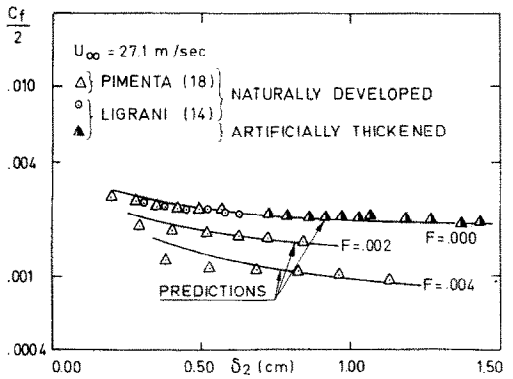

FIG. 3. Prediction of skin friction coefficients in boundary layers developing over uniform spheres roughness.

PREDICTION RESULTS

Figures 1 and 2 show pipe flow predictions of hydrodynamic data from Nikuradse [17] and heat transfer data from Dipprey and Sabersky [15], respectively. In Fig. 1, for values of R/k_s ranging from 15.0 to 126.0, the predictions show excellent agreement with the data. In Fig. 2, data for $k_s/D = 0.448$ at three different values of the molecular Prandtl number over a range of Re_D are well represented by predictions.

Predictions of Stanford rough-wall turbulent boundary layer $C_f/2$ and St data with and without transpiration are shown in Figs. 3 and 4 for a freestream velocity of 27.1 m s^{-1}. The roughness for these experimental studies (Pimenta, Moffat and Kays [18] and Ligrani, Moffat and Kays [14]) consists of uniform spheres, where each sphere is 1.27 mm in diameter, which is equivalent to a sandgrain roughness height of 0.79 mm. The data for both Figs. 3 and 4 show excellent agreement with predictions. Equally good agreement between data and predictions exists at higher fully rough velocities, and for lower velocities when the boundary layers

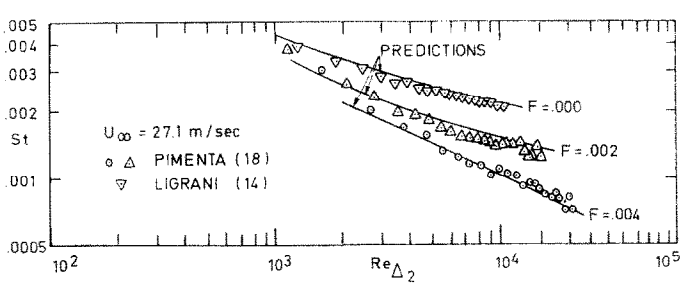

FIG. 4. Prediction of Stanton numbers in boundary layers developing over uniform spheres roughness.

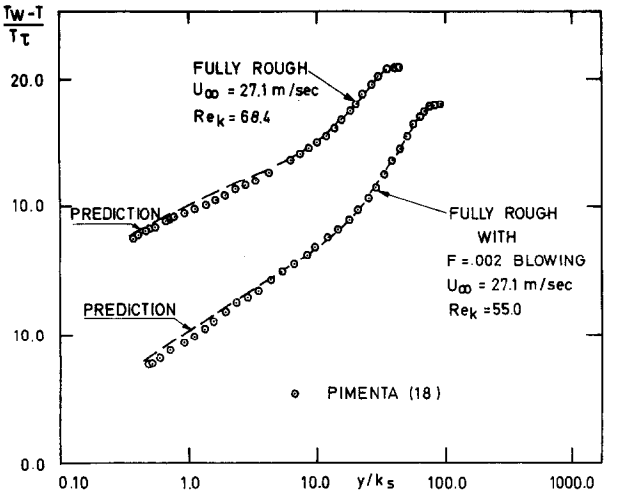

FIG. 5. Prediction of mean temperature profiles in boundary layers developing over uniform spheres roughness.

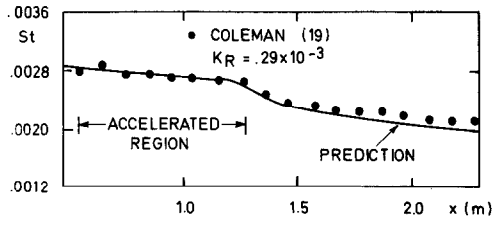

FIG. 6. Prediction of Stanton numbers in an accelerated fully rough boundary layer developing over uniform spheres roughness.

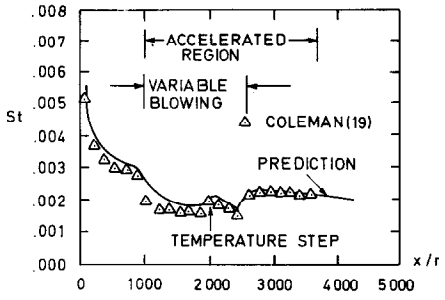

FIG. 7. Prediction of Stanton numbers in a fully rough boundary layer developing over uniform spheres roughness with acceleration, steps in blowing, variable blowing, and a wall temperature step.

are transitionally rough. Fully rough temperature profiles with and without blowing are compared with predictions in Fig. 5. The quality of the agreement shown in Fig. 5 is the same for mean velocity profiles.

A comparison is made in Fig. 6 between predictions and data for a flow over uniform spheres roughness with acceleration (Coleman, Moffat and Kays [19]). The magnitude of the fully rough acceleration parameter K_R for the flow is 0.29 $\times 10^{-3}$. In Fig. 7, predictions are compared to measurements in a flow over the same type of roughness with acceleration, steps in blowing, variable blowing, and a wall temperature step. For both cases, the agreement between the predicted and measured Stanton numbers is very good, considering the complicated nature of the boundary conditions imposed on the flows.

CONCLUSIONS

Skin friction coefficients, Stanton numbers, mean velocity profiles and mean temperature profiles are predicted for boundary layers developing over uniform spheres roughness with and without favourable pressure gradient, and with and without transpiration. The wall scalar properties are also predicted for flow in pipes with closely packed sandgrain type roughness. The near-wall mixing-length equation used for the predictions is used in conjunction with a wall temperature step and a turbulent Prandtl number distribution, along with the same mixing-length equations for the outer regions of smooth wall boundary layers.

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A NUMERICAL METHOD FOR THE SOLIDIFICATION OF A BINARY ALLOY

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NOMENCLATURE

c, normalized concentration;
$\hat{c}$, heat capacity, $[J\,°C\,kg^{-1}]$;
D, mass diffusivity, $[m^2\,s^{-1}]$;
k, thermal conductivity, $[J\,°C\,m^{-1}\,s^{-1}]$;
L, length of slab, [m];
R, defined by equation (11);
r, mass flux, Dc';
S, defined by equation (12);
s, location of the phase front;
t, time variable, [s];
u, temperature, [°C];
u^*, interface temperature, [°C];
v, heat flux, ku';
w, defined by equation (13);
x, length variable, [m];
z, defined by equation (14).

Greek symbols

α, boundary temperature, [°C];
β, boundary concentration;
Δt, time step, [s];
λ, heat of fusion, $[J\,kg^{-1}]$;
ρ, mass density, $[kg\,m^{-3}]$.

Subscripts

$+$, indicates liquid phase;
$-$, indicates solid phase;
0, indicates initial time;
$n-1$, indicates time level $n-1$.

Superscripts

$'$, d/dx.

1. INTRODUCTION

IT IS the purpose of this communication to introduce a numerical method for the solidification of a one-dimensional binary alloy. The method is a straightforward extension of the technique described in [4] for the two-phase Stefan problem. It is applied to the heat and mass balance equations and specifically tracks the phase front. The method has several useful features:

(1) It applies to the primary variables of temperature and solute concentration.
(2) It permits solute diffusion both in the liquid and solid phase.
(3) General phase diagrams are acceptable for the liquid–solid phase change.
(4) The method is applicable to systems with heat and concentration dependent diffusion parameters.
(5) The method is time implicit and can cope with discontinuous systems as well as with the vastly different time constants for the heat and mass diffusion. Last, but not least the method is straightforward to implement and cheap to run.